

Gaussian Fields and Random Packing

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Consider sequential packing of unit volume balls in a large cube, in any dimension and with Poisson input. We show after suitable rescaling that the spatial distribution of packed balls tends to that of a Gaussian field in the thermodynamic limit. The results cover related applied models, including ballistic deposition and spatial birth-growth models.

KEY WORDS: Gaussian fields; random packing; ballistic deposition; spatial birth-growth models.

1. INTRODUCTION

The following prototype random packing model is known as the basic Random Sequential Adsorption Model (RSA) for hard spheres on a continuum surface. Open balls $B_{1,n}, B_{2,n}, \dots$, of unit radius arrive sequentially and uniformly at random in the d -dimensional cube Q_n having volume n and centered at the origin. Let the first ball $B_{1,n}$ be *packed*, and recursively for $i = 2, 3, \dots$, let the i th ball $B_{i,n}$ be packed iff $B_{i,n}$ does not overlap any ball in $B_{1,n}, \dots, B_{i-1,n}$ which has already been packed. If not packed, the i th ball is discarded. Given a positive integer k , let $N(\{B_{1,n}, \dots, B_{k,n}\})$ be the number of balls packed out of the first k arrivals. $N_{n,d}(k) := N(\{B_{1,n}, \dots, B_{k,n}\})$ are called random packing numbers.

Lattice packing is defined analogously to continuum packing, save for the obvious constraint that centers of incoming balls are constrained to lie on a lattice.

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There is a vast literature involving versions of the RSA model on continuum and lattice substrates. There is a plethora of experimental results and a notable dearth of mathematically rigorous results, particularly in more than one dimension. For surveys of the extensive literature, see Evans,⁽⁹⁾ Senger *et al.*,⁽²³⁾ Bartelt and Privman,⁽³⁾ Adamczyk *et al.*,⁽¹⁾ Talbot *et al.*,⁽²⁶⁾ and ref. 21.

In addition to their fundamental role in adsorption modelling, sequential packing models arise in the study of hard core interactions in physical and materials science, spatial growth models in crystallography and biology (Evans,⁽⁹⁾ Section III, Garcia-Ruiz *et al.*⁽¹⁰⁾), and in the study of polymer reactions.^(10, 21) In modelling communication protocols (Coffman *et al.*⁽⁷⁾), RSA is called *on-line packing*.

Consider input of size $\mathcal{P}(\tau n)$, where $\mathcal{P}(\tau n), n \geq 1$, are independent Poisson random variables with parameter τn . Thus the deposition intensity is a Poisson random variable with parameter τ . Deposition intensity in the continuum is the average number of particles arriving per unit volume, whereas in the lattice it is the average number of arriving particles per lattice point. In ref. 20, the authors show that for both continuum and lattice packing, the Poisson packing numbers $N_{n,d}(\mathcal{P}(\tau n))$ satisfy a thermodynamic limit as well as a central limit theorem.

Theorem 1.1 (LLN and CLT for packing numbers (ref. 20)). For all $\tau \in (0, \infty)$ and all $d \geq 1$ there are positive constants $\alpha_{d,\tau}$ and $\eta_{d,\tau}$ such that

$$\frac{N_{n,d}(\mathcal{P}(\tau n))}{n} \rightarrow \alpha_{d,\tau} \quad \text{c.m.c.c.} \quad (1.1)$$

while

$$\frac{N_{n,d}(\mathcal{P}(\tau n)) - EN_{n,d}(\mathcal{P}(\tau n))}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \eta_{d,\tau}^2) \quad (1.2)$$

and

$$n^{-1} \text{Var } N_{n,d}(\mathcal{P}[\tau n]) \rightarrow \eta_{d,\tau}^2. \quad (1.3)$$

Here c.m.c.c. denotes convergence of means and complete convergence, $\alpha_{d,\tau}$ is called the coverage function, and $\mathcal{N}(0, \sigma^2)$ denotes a normal random variable with mean zero and variance σ^2 . The limits (1.1)–(1.3) also hold if the Poisson packing numbers $N_{n,d}(\mathcal{P}(\tau n))$ are replaced by the finite input packing numbers $N_{n,d}([\tau n])$.⁽²⁰⁾

Our goal here is to investigate the large n *distribution* of packed balls on the d -dimensional substrate \mathcal{Q}_n . Even in dimension $d = 1$ little is known about the distribution of the point process on \mathbb{R} induced by the packed balls. Our main result is that the basic RSA packing process on $\mathcal{Q}_n \subset \mathbb{R}^d$ induces a point measure which, when suitably rescaled, converges to that of a Gaussian field on \mathbb{R}^d in the thermodynamic limit. The CLT given by (1.2) of Theorem 1.1 is a by-product. Our main result also holds for variants of the basic packing process.

The packing process is defined on the infinite substrate \mathbb{R}^d in a natural way (see, e.g., ref. 16 for details). Represent the centers of the incoming balls, together with their arrival times, as a point set in $\mathbb{R}^d \times \mathbb{R}^+$. Points of $\mathbb{R}^d \times \mathbb{R}^+$ are generically denoted by $w := (x, t_x)$, where $x \in \mathbb{R}^d$, $t_x \in \mathbb{R}^+$. Given a locally finite point set $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}^+$, we let $\alpha(\mathcal{X})$ denote the subset of \mathcal{X} which is accepted in the packing process. We let $\pi(\mathcal{X}) \subset \mathbb{R}^d$ denote the projection of $\alpha(\mathcal{X})$ onto \mathbb{R}^d . $\pi(\mathcal{X})$ is the point process on the substrate \mathbb{R}^d formed by the accepted balls.

Let \mathcal{P}_τ denote a rate one homogeneous Poisson point process on $\mathbb{R}^d \times [0, \tau]$. In the lattice setting, \mathcal{Q} denotes a collection of rate one homogeneous Poisson point processes on \mathbb{R} indexed by the lattice points \mathbb{Z}^d and embedded in a natural way into $\mathbb{R}^d \times \mathbb{R}^+$. \mathcal{P}_τ and \mathcal{Q} henceforth represent the input for the continuum and lattice packing processes, respectively. The processes $\pi(\mathcal{P}_\tau)$ and $\pi(\mathcal{Q})$ are thinned Poisson point processes; we will be interested in the distribution of these measures.

For any set $A \subset \mathbb{R}^d$, let $\mathcal{P}_{\tau,A}$ be the restriction of \mathcal{P}_τ to $A \times [0, \tau]$ and let \mathcal{Q}_A denote the restriction of \mathcal{Q} to A . For any Borel set $B \subset \mathbb{R}^d$ define the random field

$$v_\tau(B) := \sum_{x \in \pi(\mathcal{P}_\tau)} \delta_x(B)$$

and for all $B \subset A$ define the random field

$$v_{\tau,A}(B) := \sum_{x \in \pi(\mathcal{P}_{\tau,A})} \delta_x(B).$$

Thus v_τ is the random field on subsets of \mathbb{R}^d induced by the packing process on \mathbb{R}^d whereas $v_{\tau,A}$ is the random field on subsets of A induced by the packing process on A . Since $\sup_B v_\tau(B) = \infty$ and $\sup_B v_{\tau,A}(B) \leq \text{volume}(A)$, we call v_τ and $v_{\tau,A}$ the *infinite volume* and *finite volume* packing measures, respectively. Edge effects show that in general these two random fields do not coincide on subsets of A .

We similarly define random fields induced by the lattice packing process as follows. For any Borel set $B \subset \mathbb{R}^d$ define

$$\mu(B) := \sum_{x \in \pi(\mathcal{Q})} \delta_x(B)$$

and

$$\mu_A(B) := \sum_{x \in \pi(\mathcal{Q}_A)} \delta_x(B).$$

For all Borel sets $B \subset \mathbb{R}^d$, consider the rescaled infinite volume continuum packing measures

$$v_{\tau, \lambda}(B) := \frac{v_{\tau}(\lambda B) - E v_{\tau}(\lambda B)}{\sqrt{\lambda^d}} \tag{1.4}$$

and for $B \subset A$, define the rescaled finite volume continuum packing measures

$$v_{\tau, A, \lambda}(B) := \frac{v_{\tau, \lambda A}(\lambda B) - E v_{\tau, \lambda A}(\lambda B)}{\sqrt{\lambda^d}}. \tag{1.5}$$

The rescaled centered lattice packing measures μ_{λ} and $\mu_{A, \lambda}$ are defined analogously. All packing measures μ (or v) considered here are spatially homogeneous in the sense that the vectors $\langle v(B_1), \dots, v(B_k) \rangle$ and $\langle v(B_1 + x), \dots, v(B_k + x) \rangle$ have the same distribution for any $x \in \mathbb{R}^d$ and any Borel subsets B_1, \dots, B_k .

Recall that a sequence of random fields $\mu_n, n \geq 1$, converges in distribution to μ if and only if all finite dimensional distributions $\langle \mu_n(B_1), \dots, \mu_n(B_k) \rangle$ converge to $\langle \mu(B_1), \dots, \mu(B_k) \rangle$, where B_1, \dots, B_k are Borel subsets of \mathbb{R}^d . We say that μ is a (generalized) Gaussian field on the Borel subsets of \mathbb{R}^d if for all Borel sets B_1, \dots, B_m , $\langle \mu(B_1), \dots, \mu(B_k) \rangle$ is a multivariate Gaussian vector. The following are our main results.

Theorem 1.2 (Infinite volume packing measures converge to a Gaussian field).

(i) (Poisson input continuum packing). For all $\tau < \infty$, $v_{\tau, \lambda}$ converges in distribution as $\lambda \rightarrow \infty$ to a Gaussian random field with covariance kernel $K_{v, \tau}$ concentrated on the diagonal, that is

$$K_{v, \tau}(x, y) := C_{v, \tau} \delta(x - y),$$

where $C_{v, \tau}$ is a constant depending on τ .

(ii) (Infinite input lattice packing). μ_λ converges in distribution as $\lambda \rightarrow \infty$ to a Gaussian random field with covariance kernel K_μ concentrated on the diagonal, that is

$$K_\mu(x, y) := C_\mu \delta(x - y),$$

where C_μ is a constant.

The next theorem clearly extends Theorem 1.1. To obtain Theorem 1.1 we simply let A be the unit cube centered at the origin of \mathbb{R}^d .

Theorem 1.3 (Finite volume packing measures converge to a Gaussian field). Let $A \subset \mathbb{R}^d$ have piecewise smooth boundary. Then:

(i) (Poisson input continuum packing). For all $\tau < \infty$, $\nu_{\tau, A, \lambda}$ converges in distribution as $\lambda \rightarrow \infty$ to a Gaussian random field with covariance kernel $K_{\nu, \tau}$.

(ii) (Infinite input lattice packing). $\mu_{A, \lambda}$ converges in distribution as $\lambda \rightarrow \infty$ to a Gaussian random field with covariance kernel K_μ .

Remarks.

(i) We will see in the sequel that

$$C_{\nu, \tau} := \int_{\mathbb{R}^d} [r_2(0, x) - r_1(0) r_1(x)] dx + r_1(0),$$

where r_1 and r_2 are the one and two point correlation functions for the spatial point process of packed points on \mathbb{R}^d . There is a similar expression for C_μ . Neither $K_{\nu, \tau}$ nor K_μ depend on the set A .

(ii) By definition of weak convergence, Theorem 1.2 tells us that for any Borel sets B_1, \dots, B_m in \mathbb{R}^d the m -vector with entries given by the $\nu_{\tau, \lambda}$ measure of B_1, \dots, B_m tends to a Gaussian limit with covariance matrix $C_{\nu, \tau}[\text{volume}(B_i \cap B_j)]$. Therefore the number of balls accepted in disjoint subsets B_1, \dots, B_m are uncorrelated (and thus independent) in the large λ limit. Theorem 1.3 makes an analogous statement for the m -vector with entries given by the $\nu_{\tau, A, \lambda}$ measure of B_1, \dots, B_m .

(iii) Proving convergence to a Gaussian field for infinite input continuum packing remains an open problem. In dimension $d = 1$, Dvoretzky and Robbins⁽⁸⁾ showed that the number of packed balls asymptotically converges to a normal random variable. Theorem 1.3(ii) adds to results of Penrose,⁽¹⁸⁾ who shows that the number of packed balls in the lattice setting satisfies a central limit theorem.

2. RELATED MODELS

There are a multitude of variants of the basic RSA packing model.^(9, 21) The approach taken here shows that the spatial distribution of packed balls (particles) for many of these variants converges to a Gaussian field. We discuss these variants below. Formal details may be found in ref. 20.

2.1. Random Shapes and Types

The basic RSA model assumes that the incoming particle is a ball of unit radius. However, this assumption may be relaxed to allow random shapes and random sets in \mathbb{R}^d , a widely considered model.⁽³⁾ More generally, arriving particles may have a random type or spin, not necessarily representing shape or size. In one dimension such a model is considered by Itoh and Shepp.⁽¹²⁾ Instead of considering the point process defined by the point set of accepted particles, we may consider the point process consisting of only those accepted particles of a particular kind or type. Theorems 1.2 and 1.3 hold for the measures induced by such point processes.

2.2. Time Dependent Models

The basic RSA packing model can be generalized to include the case in which a packed ball remains in place for a random period of time at the end of which it is removed, i.e., desorbs. This is a dynamic model, which among other things, describes the reversible deposition of particles on substrates.^(23, 26) Assuming that the spatial locations and arrival times of particles are given by a space-time Poisson process of unit intensity on $\mathbb{R}^d \times [0, \tau]$, the contribution of a particle to the point process of adsorbed points is determined not only by whether it is accepted or not, but also by whether, if accepted, it desorbs by time τ . The point measures given by the process of adsorbed points are defined analogously to (1.4), (1.5) and in the large λ limit satisfy the Gaussian structure results of Theorems 1.2 and 1.3.

We can also extend the basic packing model to a generalized version of the classical birth-growth model on \mathbb{R}^d in which cells are formed at random locations $x_i \in \mathbb{R}^d$ at times $t_i, i = 1, 2, \dots$ according to a unit intensity homogeneous space-time Poisson point process. When a new cell is formed, its center x_i is called its “seed.” Once a seed is born, the cell around it initially takes the form of a ball of possibly random radius. If the initial radii of cells are zero a.s., then the model is referred to as the Johnson–Mehl model (see Stoyan and Stoyan⁽²⁵⁾). The ball immediately generates a cell by growing radially in all directions with a constant speed. Whenever one cell touches another, it stops growing in that direction. New seeds, and

the cells around them, form only in the uncovered space in \mathbb{R}^d . The point measures given by the process of seeds are defined analogously to (1.4), (1.5) and *in the large λ limit, they satisfy the Gaussian limit results of Theorems 1.2 and 1.3*. This adds to results of Chiu and Quine,⁽⁶⁾ who study the restriction of the total number of seeds generated by the stationary birth-growth model on \mathbb{R}^d to a large window of \mathbb{R}^d . Chiu and Quine⁽⁶⁾ consider only the number of seeds generated and not their spatial distribution. Convergence to a Gaussian field of the point measure of accepted seeds also adds to results of Penrose⁽¹⁸⁾ and Penrose and Yukich⁽²⁰⁾ who prove CLT's for the total number of accepted seeds in the infinite and finite input settings, respectively.

2.3. Ballistic Deposition Models

The standard random particle deposition model considers random size i.i.d. $(d+1)$ -dimensional balls ("particles") which rain down sequentially at random onto a d dimensional substrate of volume n , and centered around the origin.

When a particle arrives on the existing agglomeration of deposited balls, the particle may slip and roll over existing particles, undergoing displacements, stopping when it reaches a position of lower height (the surface relaxation (SR) model⁽²⁾). If a particle reaches the surface \mathbb{R}^d , it is irreversibly fixed on it; otherwise, the particle is removed from the system and the next sequenced particle is considered. The rolling process does not displace already deposited particles; there is no updating of existing particles. For $d=1$, the model dates back to Solomon.⁽²⁴⁾ Senger *et al.*⁽²³⁾ describe the many experimental results.

The accepted particles all lie on the substrate and are represented by points in \mathbb{R}^d . The position of an accepted particle is a translate of the original location in \mathbb{R}^d above which it originally comes in. The point process of accepted particles defines a point measure on \mathbb{R}^d . The rescaled measures are defined analogously to (1.4) and (1.5). The methods described below may be easily modified to show that in the large λ limit, *these measures converge to a Gaussian random field*. This thus generalizes the results of ref. 20 (Theorem 2.2(b)), which only considers the distribution of the total number of accepted particles. It also adds to results of Penrose,⁽¹⁷⁾ who proves a CLT for the total number of accepted particles with infinite input.

3. AUXILIARY RESULTS

Throughout we let $W := V \times \mathbb{R}^+$, where $V := \mathbb{R}^d$. Recall that points of W are generically denoted by $w := (x, t_x)$, where $x \in V$, $t_x \in \mathbb{R}^+$. The spatial

location of the incoming balls, together with their arrival times, are represented by points in W . The interaction range of arriving balls is just the common diameter of the balls, that is to say equals two. Throughout, $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d .

3.1. Exponential Decay of Interactions

Let $\mathcal{X} \subset W$ be a locally finite point process. Following⁽²⁰⁾ we make \mathcal{X} into the vertex set of an *oriented graph* by including an edge from $w_1 := (x_1, t_{x_1})$ to $w_2 := (x_2, t_{x_2})$ whenever $\|x_1 - x_2\| \leq 2$ and $t_{x_1} \leq t_{x_2}$. Given $w \in \mathcal{X}$, let $A_{\text{out}}(w, \mathcal{X})$ be the set of points (forward cone) in \mathcal{X} that can be reached from w by a directed path in this graph (along with w itself). Let $A_{\text{in}}(w, \mathcal{X})$ be the set of points (backward cone) in \mathcal{X} from which the point w can be reached by a directed path in this graph (along with w itself). Finally, consider the ‘‘causal cone’’

$$A_{\text{out, in}}(w, \mathcal{X}) := A_{\text{out}}(w, \mathcal{X}) \cup A_{\text{in}}(w, \mathcal{X}).$$

The next result justifies describing the sets $A_{\text{out, in}}(w, \mathcal{P}_\tau)$ and $A_{\text{out, in}}(w, \mathcal{Q})$ as ‘‘cones.’’

Lemma 3.1. Fix τ and let \mathcal{X} be either \mathcal{P}_τ or \mathcal{Q} . There exist positive constants $\gamma := \gamma(\tau)$, $\beta := \beta(\tau)$, and $\rho := \rho(\tau)$, such that the causal cone associated with $w := (x, t_x) \in \mathcal{X}$ belongs to the set

$$C_R(w) := \{(y, t_y) : \|x - y\| \leq \beta |t_x - t_y| + R\}$$

with probability at least $1 - \rho \exp(-\gamma R)$.

Proof. For the case that \mathcal{X} is the finite input set \mathcal{P}_τ , this result is just Lemma 4.2 of ref. 20. For the case that \mathcal{X} is \mathcal{Q} , we will use an ‘‘invasion percolation’’ argument similar to that in ref. 20. The following argument actually also holds if \mathcal{X} is the infinite continuum input \mathcal{P} . First, we discretize the space V by tiling it by lattice cubes of sufficiently large size so that no ball can intersect non-adjacent cubes. More precisely, we choose a full rank lattice $L \subset V$ (a sublattice of \mathbb{Z}^d in the lattice case) with the elementary cube C , $V = \bigcup_{\mathbf{i} \in L} C_{\mathbf{i}}$, where $C_{\mathbf{i}} := C + \mathbf{i}$ and the size of the cube C is chosen as to satisfy $C + B \subset 2C$ (a further restriction in the lattice case is that the boundary of C does not contain any point with integer coordinates, to avoid ambiguity).

For each cube $C_{\mathbf{i}}$, $\mathbf{i} \in L$, consider the arrival times of the points whose centers belong to this cube; they form a family $P(\mathbf{i})$, $\mathbf{i} \in L$, of independent Poisson point processes.

Now, fix a point $w = (x, t_x)$ and assume, without loss of generality, that x belongs to the cube centered at the origin. Let us restrict attention to the *forward* cone and, to simplify notation, assume that $t_x = 0$, i.e., $w = (x, 0)$. Call a *path* $\pi := \pi_0, \pi_1, \dots, \pi_p$ (of length $p := |\pi|$) a collection of points in L , such that for any two consecutive points π_k and π_{k+1} , the cubes C_{π_k} and $C_{\pi_{k+1}}$ share at least one vertex. For a path π of length p , call the increasing sequence of arrival times $0 = t_0 < \dots < t_p$ *admissible* if $t_k \in P(\pi_k)$ for all $0 \leq k \leq p$. Note that any path π admits many admissible sequences. Given a path π of length p , let $T(\pi)$ be the infimum of the terminal values t_p .

The relations above imply that if the forward cone of $(x, 0)$ does not belong to $C_R(w)$, then for some \mathbf{i} there exists a path π from $\mathbf{0}$ to \mathbf{i} such that $|\mathbf{i}| > \beta T(\pi) + R$, that is,

$$T(\pi) < (|\mathbf{i}| - R) / \beta.$$

The probability that such a path exists is majorized by the sum over all possible paths π , starting at the origin, of the probability

$$P[T(\pi) < (|\pi_{|\pi|}| - R) / \beta] \leq P[T(\pi) < (k|\pi| - R) / \beta]$$

(we use here the obvious fact that the distance between the last point of a path of length p and the origin is at most kp , where k is a constant depending only on the lattice L and the dimension d).

The last piece we need before starting calculations is the fact that for a path π of length p , $T(\pi)$ is simply the sum of p i.i.d. exponential variables Y . Choose θ large enough so that $E \exp(-\theta Y) \leq 3^{-(d+2)}$ and choose β large enough so that $e^{\theta k/\beta} \leq 3$. Let $a := (kp - R) / \beta$. Then

$$P(\pi) := P[T(\pi) < a] \leq e^{\theta a} (E[e^{-\theta Y}])^p.$$

By definition of a and the choice of θ and β the above is bounded by

$$e^{\theta kp/\beta} 3^{-p(d+2)} \leq 3^{-p(d+1)}.$$

Since the number of paths starting at $\mathbf{0}$ of length p is bounded by 3^{dp} , the probability that the forward cone of $(x, 0)$ does not belong to $C_R(w)$ is bounded above by the sum of the terms $P(\pi)$ over all possible paths π starting at $\mathbf{0}$ and of length at least R/k . This probability is thus bounded by

$$\sum_{p \geq R/k} 3^{dp} P[T(\pi) < a] \leq \sum_{p \geq R/k} 3^{dp} 3^{-p(d+1)},$$

which gives the desired exponential decay in R .

Since the case of the backward cone may be handled by using a “time-reversal,” this completes the proof of Lemma 3.1. ■

Let $\mathcal{X} \subset W$ be a point set. In keeping with the terminology of statistical mechanics, we write $\sigma(w, \mathcal{X}) = 1$, if the ball centered at w is accepted with respect to \mathcal{X} and 0 otherwise. A trivial but useful observation is that if $A_{\text{in}}(x, \mathcal{X}_1) = A_{\text{in}}(x, \mathcal{X}_2)$, then $\sigma(x, \mathcal{X}_1) = \sigma(x, \mathcal{X}_2)$.

3.2. Correlation Functions for the Process of Accepted Points

To establish convergence of random point measures to a Gaussian field, one may employ the method of moments,⁽¹¹⁾ which depends heavily on the use of correlation functions.

Let \mathcal{P} denote \mathcal{P}_τ or \mathcal{Q} . Recall that $\alpha(\mathcal{P})$ is the subset of \mathcal{P} which is accepted in the packing process. Let $1_B(\alpha(\mathcal{P})) = 1$ if $\alpha(\mathcal{P}) \cap B$ is non-empty, and let $1_B(\alpha(\mathcal{P})) = 0$ otherwise. Given $w_1, \dots, w_k \in W$, the k -point correlation functions⁽²²⁾ of the point process $\alpha(\mathcal{P})$ of accepted points are defined as

$$r_k(w_1, \dots, w_k) := \lim_{\varepsilon_1 \rightarrow 0, \dots, \varepsilon_k \rightarrow 0} \frac{E_{\mathcal{P}}[1_{(w_1+\varepsilon_1\Omega)}\alpha(\mathcal{P}) \cdots 1_{(w_k+\varepsilon_k\Omega)}\alpha(\mathcal{P})]}{\varepsilon_1^{d+1} \cdots \varepsilon_k^{d+1} \omega^k}, \tag{3.1}$$

where $w_i \neq w_j$, $1 \leq i \neq j \leq k$, Ω is the ball in W of unit radius and ω is its volume.

To clarify the nature and existence of these correlation functions we introduce for $w_1, \dots, w_k \in W$ the functions

$$\bar{r}_k(w_1, \dots, w_k) := E_{\mathcal{P}} \left[\prod_{i=1}^k \sigma(w_i, \mathcal{P}) \right].$$

The functions \bar{r}_k are the probabilities that all points $\{w_1, \dots, w_k\}$ are packed with respect to an independent sample from \mathcal{P} . We notice here that this probability is always positive (unless the balls centered at the points x_1, \dots, x_k themselves intersect), regardless of the heights t_{x_i} of the points.

We want to show that \bar{r}_k are continuous in the arguments $w_i := (x_i, t_{x_i})$, $1 \leq i \leq k$. Let $B_k \subset W^k$ be the set of tuples w_1, \dots, w_k such that $|x_i - x_j| < 1$ for some $1 \leq i \leq j \leq k$. Then $\bar{r}_k = 0$ on B_k . We will show the continuity of \bar{r}_k on the complement of B_k when $\mathcal{P} = \mathcal{P}_\tau$; continuity in the lattice setting follows from the arguments below and is easier since it only involves showing continuity in the time coordinate.

Since

$$|\bar{r}_k(w_1, \dots, w_k) - \bar{r}_k(w'_1, \dots, w'_k)|$$

is bounded by

$$\sum_{i=1}^{k-1} |\bar{r}_k(w'_1, \dots, w'_{i-1}, w_i, \dots, w_k) - \bar{r}_k(w'_1, \dots, w'_i, w_{i+1}, \dots, w_k)|, \quad (3.2)$$

it will suffice to show that \bar{r}_k is continuous in each of its k arguments. We will show

$$|\bar{r}_k(w_1, \dots, w_k) - \bar{r}_k(w'_1, \dots, w'_k)| \leq C |w_1 - w'_1|.$$

The other summands in (3.2) are bounded similarly. For all $x \in V$, let $B(x)$ denote the ball in V of unit radius centered at x . $B(w)$, $w \in W$, is defined similarly. Without loss of generality, assume that $|x_1 - x'_1| < 1$, so that $B(x_1)$ and $B(x'_1)$ overlap.

Consider the event $\prod_{i=1}^k \sigma(w_i, \mathcal{P}) \neq \sigma(w'_1, \mathcal{P}) \prod_{i=2}^k \sigma(w_i, \mathcal{P})$. If this event occurs, then either the ball $B(x_1)$ is packed and $B(x'_1)$ is not packed or vice versa. This means that the oriented graphs on $\mathcal{P} \cup \{w_1, \dots, w_k\}$ and $\mathcal{P} \cup \{w'_1, \dots, w_k\}$ are different. The only way that this can happen is if \mathcal{P} satisfies at least one of the following two conditions (assume without loss of generality that $t_{x_1} \leq t_{x'_1}$): (a) the cylinder set

$$[B(x_1) \Delta B(x'_1)] \times [0, t_{x'_1}]$$

intersects at least one ball $B(w)$, $w \in \mathcal{P}$, whereas the cylinder set

$$[B(x_1) \cap B(x'_1)] \times [0, t_{x_1}]$$

does not intersect any ball $B(w)$, $w \in \mathcal{P}$, or (b) the strip

$$[B(x_1) \cup B(x'_1)] \times [t_{x_1}, t_{x'_1}]$$

intersects at least one ball $B(w)$, $w \in \mathcal{P}$.

Condition (a) implies that

(i) the cylinder centered at $(x_1 + x'_1)/2$ with radius $2 - |x_1 - x'_1|/2$ and height t_{x_1} does not contain any point from \mathcal{P} , whereas

(ii) the cylinder centered at $(x_1 + x'_1)/2$ with radius $2 + |x_1 - x'_1|/2$ and height $t_{x'_1}$ contains a point from \mathcal{P} .

Conditions (i) and (ii) imply that there is a set having volume of order $|x_1 - x'_1|$ and containing at least one point from \mathcal{P} . Similarly, condition (b) implies that there is a set having volume of order $|t_{x_1} - t_{x'_1}|$ and containing

at least one point from \mathcal{P} . The probability that a subset of W of small volume contains at least one point from \mathcal{P} is of the order of the volume. It follows that the events (a) and (b) have probability of order $|x_1 - x'_1| + |t_{x_1} - t_{x'_1}|$, showing that the functions \bar{r}_k are continuous on the complement of B_k .

Rewrite the correlation functions r_k as

$$\lim_{\varepsilon_1, \dots, \varepsilon_k \rightarrow 0} E_{\mathcal{P}} \left[\prod_{i=1}^k 1_{w_i + \varepsilon_i \Omega}(\alpha(\mathcal{P})) \mid \prod_{i=1}^k 1_{w_i + \varepsilon_i \Omega}(\mathcal{P}) \right] \times \frac{E_{\mathcal{P}}[\prod_{i=1}^k 1_{w_i + \varepsilon_i \Omega}(\mathcal{P})]}{\prod_{i=1}^k \varepsilon_i^{d+1} \omega^k}.$$

We discard the event that any of the small balls $\{w_i + \varepsilon_i \Omega\}_{i=1}^k$ contains more than one point of \mathcal{P} as an event of lower order magnitude. The first factor in the product is just \bar{r}_k averaged over the possible positions of these solitary points in the small balls $w_i + \varepsilon_i \Omega$. Continuity of \bar{r}_k implies that, up to negligible terms, one can replace this factor by the value of \bar{r}_k at (w_1, \dots, w_k) . The second factor in the product tends, obviously, to

$$\prod_{i=1}^k h(w_i),$$

where $h(w) = h(t)$ is just the density of the underlying Poisson point process \mathcal{P} with respect to the volume form on W (in our situation, $h(t)$ is simply the indicator function of the interval $[0, \tau]$).

Summarizing, given $w_1, \dots, w_k \in W$, the correlation functions are given by the following explicit formula

$$r_k(w_1, \dots, w_k) = \bar{r}_k(w_1, \dots, w_k) \prod_{i=1}^k h(w_i). \quad (3.3)$$

3.3. The Moment Method

If $\alpha(\mathcal{P}_\tau)(B)$ represents the number of points which are accepted in the set $B \subset \mathbb{R}^d \times \mathbb{R}^+$, then its moments and variance are expressed in terms of the one and two point correlation functions via the well known formulas (Section 14.4 of Stoyan and Stoyan⁽²⁵⁾)

$$E[\alpha(\mathcal{P}_\tau)(B)] = \int_B r_1(w) dw, \quad (3.4)$$

$$E[\alpha^2(\mathcal{P}_\tau)(B)] = \int_{B \times B} r_2(w_1, w_2) dw_1 dw_2 + \int_B r_1(w_1) dw_1, \quad (3.5)$$

and

$$\text{Var}[\alpha(\mathcal{P}_\tau)(B)] = \int_{B \times B} r_2(w_1, w_2) - r_1(w_1) r_1(w_2) dw_1 dw_2 + \int_B r_1(w_1) dw_1. \tag{3.6}$$

Central to the method of moments is the notion of exponentially clustering, described as follows.

Definition 3.1. A family of functions $\{r_j: V^j \rightarrow \mathbb{R}\}_{j=1}^\infty$ exponentially clusters if for all $k, l \in \mathbb{N}^+$, there exists positive constants $A_{k,l}$ and $C_{k,l}$ such that one has uniformly

$$|r_{k+l}(w_1, \dots, w_k, w'_1, \dots, w'_l) - r_k(w_1, \dots, w_k) r_l(w'_1, \dots, w'_l)| \leq A_{k,l} \exp(-C_{k,l}d(w, w')),$$

where $d(w, w')$ is the distance between the sets $\{w_1, \dots, w_k\}$ and $\{w'_1, \dots, w'_l\}$, that is the minimum of the pairwise distances $|w_i - w'_j|$.

Exponential clustering is also known as weak exponential decrease of correlations,⁽¹⁴⁾ or simply exponential decay of correlations. Its relevance to central limit theorems is as follows.

Let ν be a spatially homogeneous point process on \mathbb{R}^d . Then ν defines a spatially homogeneous point process ν_L on the lattice \mathbb{Z}^d via

$$\nu_L := \sum_{x \in \mathbb{Z}^d} \nu(Q_x) \delta_x,$$

where Q_x is the unit cube centered at x . Exponential clustering of the correlation functions of ν implies the exponential clustering of the correlation functions for ν_L . The methods of Malyshev,⁽¹⁴⁾ who restricts attention to probability measures on $\{-1, 1\}^{\mathbb{Z}^d}$, show that the normalized lattice measures

$$\nu_{L,\lambda}(B) := \frac{\nu_L(\lambda B) - E\nu_L(\lambda B)}{\sqrt{\lambda^d}},$$

B a Borel set, converge in distribution to a Gaussian field⁽¹⁴⁾ on the Borel sets of \mathbb{R}^d (see Iagolnitzer and Souillard,⁽¹¹⁾ especially p. 576). Suppose that the lattice measures $\nu_{L,\lambda}$ approximate the normalized continuum measures

$$\nu_\lambda(B) := \frac{\nu(\lambda B) - E\nu(\lambda B)}{\sqrt{\lambda^d}}$$

in that the difference satisfies

$$\text{Var}[v(\lambda B) - v_L(\lambda B)] = O(\lambda^{d-1}) \quad (3.7)$$

for all sets B in the collection of d -dimensional rectangles \mathcal{R}_d . Then the continuum measures v_λ converge to a Gaussian field over the collection \mathcal{R}_d and thus v_λ converges in distribution to a Gaussian field (Billingsley,⁽⁵⁾ pp. 14–15).

Combining with the expression for the variance (3.6), we obtain the following result, the continuum analog of Malyshev's CLT⁽¹⁴⁾ for Gibbsian random fields.

Theorem 3.1 (Gaussian CLT). Let v be a spatially homogeneous point process such that its correlation functions exponentially cluster. Suppose the rescaled centered measures $v_\lambda(B)$ approximate the lattice measures $v_L(\lambda B)$ as in (3.7). Then, as $\lambda \rightarrow \infty$, v_λ converges in distribution to a Gaussian random field with covariance kernel

$$K(x, y) := C\delta(x - y),$$

where

$$C := \int_V r_2(0, x) - r_1(0) r_1(x) dx + r_1(0).$$

Next we show how to use Theorem 3.1 to prove Theorem 1.2.

3.4. Proof of Theorem 1.2

Recall that \mathcal{P} denotes either \mathcal{P}_τ or \mathcal{Q} . To show Theorem 1.2, we will show that the correlation functions r_k , $k \geq 1$, of the point process $\alpha(\mathcal{P})$ cluster exponentially. This implies that the correlation functions r_k^π of the point process $\pi(\mathcal{P})$ cluster exponentially, thus yielding Theorem 1.2 via Theorem 3.1.

To show exponential clustering of the r_k , $k \geq 1$, we first establish that $\alpha(\mathcal{P})$ is localized near V . This localization is of course obvious in the off-lattice case, as the t -support of the process is the bounded interval $[0, \tau]$. We cannot remove this cut-off as the current way of proof needs a rather rapid decay of the correlation functions with t which is lacking in the off-lattice case. Indeed, we have some basic estimates on bounds for the correlation functions. For example, the decay of r_1 is polynomial: in dimension

$d = 1$ it follows directly from the Rényi formula (see e.g., (2) in ref. 7) that $r_1(x, t) \sim t^{-2}$ and in higher dimensions r_1 can be also shown to be at least of order $t^{-\frac{1}{d}-1}$. The next lemma shows that the correlation functions decay exponentially in t in the finite input off-lattice case as well as in the infinite input lattice setting.

Proposition 3.1. The correlation functions $r_k, k \geq 1$, of the point process $\alpha(\mathcal{P})$ decay exponentially with t 's, i.e., for any k there exists positive constants A_k and C_k such that for $w_1, \dots, w_k \in W$ we have

$$r_k(w_1, \dots, w_k) \leq A_k \exp(-C_k \max_i (t_i)).$$

Proof. Let $w_i = (x_i, t_i), 1 \leq i \leq k$. Consider first the correlation functions for $\alpha(\mathcal{P}_\tau)$. Proposition 3.1 is clearly satisfied as in this case the left hand side of the inequality vanishes for $\max_{i \leq k} t_i \geq \tau$.

Now consider the correlation functions for $\alpha(\mathcal{Q})$, that is the correlation functions for the point process of accepted points on the lattice. Notice that, obviously,

$$\bar{r}_k(w_1, \dots, w_k) \leq \bar{r}_1(w_i)$$

for any $1 \leq i \leq k$. Further, $\bar{r}_1(w_1)$ is just the probability that the point $w_1 = (x, t)$ is packed with respect to \mathcal{Q} . However, this implies that all points $w_s = (x, st), 0 < s < 1$, can be packed with respect to \mathcal{Q} , and therefore, that none of the points $w_s, s < 1$, are present in the sample \mathcal{Q} . This has probability $\exp(-at)$, where $a > 0$ is the intensity measure of the interval $\{(x, s), 0 \leq s \leq 1\}$. Thus $r_1 = \bar{r}_1$ decays exponentially with t and Proposition 3.1 is proved in the lattice setting. ■

Proposition 3.2. Given points $\{w_1, \dots, w_k\}$ and $\{w'_1, \dots, w'_l\}$, let d denote the minimum of the pairwise distances $|w_i - w'_j|$. Let $E_{k,l} := E_{k,l}(w_1, \dots, w_k, w'_1, \dots, w'_l; \mathcal{P})$ be the event that the backward cone of each point w_i (respectively w'_j) with respect to \mathcal{P} belongs to the cylinder of radius $d/2$ centered at x_i (respectively x'_j). Then

$$|r_{k+l}(w_1, \dots, w_k, w'_1, \dots, w'_l) - r_k(w_1, \dots, w_k) r_l(w'_1, \dots, w'_l)| \leq C_{k,l} (P[E_{k,l}^c])^{1/2}. \quad (3.8)$$

Proof. Define for any event E measurable with respect to the sigma algebra generated by \mathcal{P}

$$\bar{r}_k^E(w_1, \dots, w_k) := E_{\mathcal{P}} \left[\prod_{i=1}^k \sigma(w_i, \mathcal{P}) 1_E \right].$$

It suffices to show that (3.8) holds with r_k replaced by \bar{r}_k . Now $|\bar{r}_k - \bar{r}_k^E|$ equals

$$E_{\mathcal{P}} \left[\prod_{i=1}^k \sigma(w_i, \mathcal{P}) 1_{E^c} \right] \leq A_k (P[E^c])^{1/2}, \tag{3.9}$$

by Cauchy–Schwarz and the boundedness of \bar{r}_k .

Observe that by the definition of $E_{k,l}$, the random variables

$$\prod_{i=1}^k \sigma(w_i, \mathcal{P}) 1_{E_{k,l}} \quad \text{and} \quad \prod_{j=1}^l \sigma(w'_j, \mathcal{P}) 1_{E_{k,l}}$$

are independent.

Hence the difference $\bar{r}_{k+l}^{E_{k,l}}(w_1, \dots, w_k, w'_1, \dots, w'_l) - \bar{r}_k^{E_{k,l}}(w_1, \dots, w_k) \times \bar{r}_l^{E_{k,l}}(w'_1, \dots, w'_l)$ vanishes and using the estimate (3.9) we obtain (3.8). ■

The clustering of the correlation functions is captured in

Proposition 3.3. The correlation functions $r_k, k \geq 1$, of the point process $\alpha(\mathcal{P})$ cluster exponentially.

Proof. We fix τ and use Lemma 3.1 describing the localization of causal cones. Let $\mathcal{P} = \mathcal{P}_\tau$; the proof for $\mathcal{P} = \mathcal{Q}$ is exactly the same. The constants γ, ρ , and β are as in Lemma 3.1. Let $w_i := (x_i, t_i), 1 \leq i \leq k$. Let $w'_j := (x'_j, t'_j), 1 \leq j \leq l$.

Consider two sets $\{w_1, \dots, w_k\}$ and $\{w'_1, \dots, w'_l\}$ at distance d . That is $d := \min |w_i - w'_j|$. We distinguish two cases.

(a) All times $t_1, \dots, t_k, t'_1, \dots, t'_l$ are less than $\frac{d}{4\beta}$. In this case with probability at least $1 - (k\rho) \exp(-\gamma d/4)$, the causal cone for each point $w_i, 1 \leq i \leq k$, belongs to the set $C_{d/4}(w_i)$ which is a subset of the cylinder of radius $d/2$ centered at x_i . The same is valid for $w'_j, 1 \leq j \leq l$. Let \mathbb{E} denote the event for which this is true. As corresponding cylinders for points from different tuples do not intersect, we conclude that \mathbb{E} implies $E_{k,l}$, where $E_{k,l}$ is defined as in Proposition 3.2. Now apply Proposition 3.2.

(b) If at least one of the times (say, t_1) is larger than $\frac{d}{4\beta}$, then by Proposition 3.1, both $r_{k+l}(w_1, \dots, w_k, w'_1, \dots, w'_l)$ and $r_k(w_1, \dots, w_k)$ decay exponentially in t_1 and therefore decay exponentially in d as well. This finishes the proof. ■

We will prove Theorem 1.2 by applying Theorem 3.1. $\pi(\mathcal{P})$ is a translationally invariant discrete (meaning that the distance between any two points is uniformly bounded from below) point process. To apply

Theorem 3.1, it suffices to show that the correlation functions cluster exponentially and that the approximation (3.7) holds.

We first verify that the correlation functions cluster exponentially. This is easy, given Propositions 3.1 and 3.3. Indeed, given $x_1, \dots, x_k \in V$, the correlation functions for $\pi(\mathcal{P})$ are given as

$$r_k^\pi(x_1, \dots, x_k) := \int_{t_1, \dots, t_k \geq 0} r_k((x_1, t_1), \dots, (x_k, t_k)) dt_1 \cdots dt_k.$$

To prove the clustering inequality for a k -tuple $\langle x_1, \dots, x_k \rangle$ and an l -tuple $\langle x'_1, \dots, x'_l \rangle$ at a distance d we split the integration domain $(\mathbb{R}^+)^k \times (\mathbb{R}^+)^l$ into two subdomains, namely $[0, d]^{k+l}$ and its complement. Since all t_i , $1 \leq i \leq k$, and t'_j , $1 \leq j \leq l$, are less than d in the first domain, we apply Proposition 3.3, using the fact that the W -distance between w_i and w'_j is at least d and the polynomial bound d^{k+l} on the volume of the integration domain. This gives a bound which is exponentially decaying with d .

In the second subdomain we apply Proposition 3.1, resulting in the estimate from above of this second part as

$$\int_d^\infty A_{V,k,l} \exp(-C_{V,k,l}s) d \operatorname{vol}(\{\max(t_1, \dots, t_k, t'_1, \dots, t'_l) \leq s\}),$$

where $A_{V,k,l} := A_k + A_l + A_{k+l}$ and $C_{V,k,l} := \min(C_k, C_l, C_{k+l})$. This is obviously exponentially decaying with d and thus the correlation functions r_k^π cluster exponentially.

It is even easier to verify the approximation (3.7). For any rectangle R , let

$$\pi(R) = \pi(\mathcal{P}) \cap R$$

and define the lattice measures by

$$\pi_L(R) = \sum_{x \in \mathbb{Z}^d} \pi(Q_x) \delta_x(R).$$

Then

$$\pi(\lambda R) - \pi_L(\lambda R) = \sum_{x \in \lambda R} \pi(\lambda R \cap Q_x) - \pi(Q_x).$$

Let R_1 denote the set of points distant at most 1 from the boundary of λR , and let r_k^R , $k \geq 1$, denote the correlation functions for $\pi(\lambda R) - \pi_L(\lambda R)$. We

notice that the summands in the above all vanish except for x belonging to some subset $R'_1 \subset R_1$. Thus using (3.6), we have

$$\begin{aligned} \text{Var}[\pi(\lambda R) - \pi_L(\lambda R)] &= \int_{R'_1 \times R'_1} r_2^R(w_1, w_2) - r_1^R(w_1) r_1^R(w_2) dw_1 dw_2 \\ &\quad + \int_{R'_1} r_1^R(w_1) dw_1. \end{aligned}$$

Following Proposition 3.3 we may show that the correlation functions r_k^R , $k \geq 1$, cluster exponentially. Thus, the integrand of the first integral decay exponentially with the distance from the diagonal. It follows that the first integral is of the order of the volume of R_1 , i.e., of the order λ^{d-1} . The second integral is also of the order λ^{d-1} since r_1^R is bounded. ■

4. PROOF OF THEOREM 1.3

We now deduce Theorem 1.3 from Theorem 1.2. We have already noted that because of edge effects, the rescaled point measures $\nu_{\tau, \lambda}$ and $\nu_{\tau, A, \lambda}$ are in general not equal. However, one can estimate the difference between the point measures $\nu_{\tau, \lambda}$ and $\nu_{\tau, A, \lambda}$. We will do this by using the exponentially clustering of the two-point correlation function to upper bound the variance of the difference.

More specifically, for an open set $A \subset \mathbb{R}^d$, let $\pi(\mathcal{P})_A := \pi(\mathcal{P}) \cap A$ and define the point measure $\pi_{\lambda A}^+ := \sum_{x \in \pi(\mathcal{P}_{\lambda A}) - \pi(\mathcal{P})_{\lambda A}} \delta_x$. $\pi_{\lambda A}^+$ is the difference between the packing process on λA and the infinite packing process restricted to λA . Similarly define $\pi_{\lambda A}^- := \sum_{x \in \pi(\mathcal{P})_{\lambda A} - \pi(\mathcal{P}_{\lambda A})} \delta_x$.

We want to estimate the variance of the number of points defining the supports of the point measures $\pi_{\lambda A}^+$ and $\pi_{\lambda A}^-$. If we could show that this variance is of order $o(\lambda^d)$, $\lambda \rightarrow \infty$, then it would follow that the random measures $\pi(\mathcal{P}_{\lambda A})$, after centering and rescaling by $\sqrt{\lambda^d}$, have asymptotically the same distribution as the centered and rescaled random measures $\pi(\mathcal{P})_{\lambda A}$, that is, have a Gaussian distribution.

To prove these variance bounds, we again resort to representation of the moments as integrals of polynomials of the correlation functions. In our case we will need just the expressions (3.4), (3.6) for the first two moments.

For all k we let r_k^\pm denote the correlation functions for $\pi_{\lambda A}^\pm$. Observe the following:

(a) The correlation function $r'_1(w) := P[w \in \pi(\mathcal{P}_{\lambda A}) | w \in \mathcal{P}_{\lambda A}]$ decays exponentially with time t for the same reasons that r_1 does.

(b) The correlation functions r_k^\pm cluster exponentially. The proof goes along the lines of the proof of Proposition 3.3, and in fact involves conditioning on the same event.

In other words, Propositions 3.1 and 3.3, together with the constants there, are valid for the point processes $\pi_{\lambda A}^\pm$.

The crucial property is the next one, which ensures that the point processes $\pi_{\lambda A}^\pm$ have support which localizes near the boundary of λA .

(c) The causal cone $A_{\text{out, in}}(w, \mathcal{P}_{\lambda A})$ of any point w with respect to the process $\mathcal{P}_{\lambda A}$ is a subset of $A_{\text{out, in}}(w, \mathcal{P}_{\lambda A})$ with respect to \mathcal{P} and coincides with it if $A_{\text{out, in}}(w, \mathcal{P})$ does not intersect the boundary $\partial(\lambda A)$.

Therefore, the correlation functions r_1^\pm of the processes $\pi_{\lambda A}^\pm$ decay exponentially with the distance of V -projection to $\partial(\lambda A)$:

$$r_1^\pm(w) \leq A_\partial \exp\{-C_\partial d(x, \partial(\lambda A))\}, \quad w := (x, t)$$

for some positive constants A_∂, C_∂ .

Together these properties imply the following estimate, which completes the proof of Theorem 1.3.

Proposition 4.1. Assume that the set A is piecewise smooth. Then the variance and the mean of the number of points in either of the processes $\pi_{\lambda A}^\pm$ is of order $O(\lambda^{d-1})$.

Proof. Start with the mean, which by (3.4), is given by

$$\int_W r_1^\pm(w) dw.$$

By properties (a) and (c) above, the correlation function r_1^\pm decays exponentially (say, as $D \exp(-Cd)$) with the distance d from $\partial(\lambda A) \subset V \times \{0\} \subset W$. Hence the integral in question is bounded above by

$$D \int_0^\infty \exp(-Cs) dB_\lambda(s),$$

where $B_\lambda(s) := \text{vol}(\{x \in \mathbb{R}^d : d(x, \partial(\lambda A)) \leq s\})$ and where $\{x \in \mathbb{R}^d : d(x, \partial(\lambda A)) \leq s\}$ is the s -neighborhood around the set λA . Since the leading term for the volume of the s -neighborhood around λA has the form $s\lambda^{d-1}C_A$, where C_A is a constant depending on the set A , the above integral has leading term in λ which is polynomial of degree $(d-1)$.

To estimate the variance, we just add the clustering condition and use the variance formula (3.6) to obtain

$$\text{Var}(\pi_{\lambda A}^+) = \int_{W \times W} [r_2^+(w_1, w_2) - r_1^+(w_1) r_1^+(w_2)] dw_1 dw_2 + \int_W r_1^+(w) dw,$$

where r_1^+ and r_2^+ are the correlation functions for the process $\pi_{\lambda A}^+$ (an analogous formula is also valid for $\text{Var}(\pi_{\lambda A}^-)$).

The second summand is just the mean, and the integrand in the first integral decays in $W \times W$ exponentially with the distance from $\partial(\lambda A) \subset W \times W$. Once again applying the integral estimates as above, we obtain $\text{Var}(\pi_{\lambda A}^+) = O(\lambda^{d-1})$. ■

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